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Guy Fayolle, Roudolph Iasnogorodski, Vadim A. A. Malyshev

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.(1) 39 63 55 11

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Guy FAYOLLE
Roudolf IASNOGORODSKI
Vadim MALYSHEV

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G. FAYOLLE[1], R. IASNOGORODSKI[2], V.A. MALYSHEV[3]

Abstract

In this paper, we caracterize the solutions of specific bi-variate functional equations. The unknown functions represent the steady state distribution of specific two-dimensional random walks on $\mathbf{Z}_+^2$. Inside the quarter plane, the jumps have amplitude one, but are arbitrary on the axes. The main result is a necessary and sufficient condition for the solutions to be algebraic, when the "group" of the random walk (associated to an algebraic curve $Q(x, y) = 0$ having genus one) is finite. The method is based on Hilbert's factorization theorems, together with a uniformisation by means of elliptic functions.

[1]INRIA - Domaine de Voluceau, Rocquencourt - BP.105 - 78153 Le Chesnay Cedex - FRANCE.

[2]Université Orléans la Source, Département de Mathématiques, Campus Universitaire - 45000 Orléans la Source - FRANCE.

[3]Moscow State University, Mechanico-Mathematical Faculty, Chair of probability, Leninskie Gori, 119899 MOSCOW - USSR.

Fonctions génératrices algébriques pour des marches aléatoires à deux dimensions

GUY FAYOLLE[1], ROUDOLF IASNOGORODSKI[2], VADIM MALYSHEV[3]

Résumé

Dans cet article on caractérise les solutions de certaines équations fonctionnelles à deux variables. Les fonctions inconnues représentent la distribution invariante de certaines marches aléatoires Z_+^2. A l'intérieur du quart de plan, les sauts sont d'amplitude 1 mais ils sont arbitraires sur les axes. Le résultat principal de l'étude est l'énoncé d'une condition nécessaire et suffisante pour que les solutions soient algébriques, quand le "groupe" de la marche aléatoire (qui est associé à une courbe algébrique $Q(x, y) = 0$ de genre un) est fini. La méthode est basée sur les théorèmes de factorisation de Hilbert, ainsi que sur une uniformisation au moyen de fonctions elliptiques.

[1]INRIA - Domaine de Voluceau, Rocquencourt - BP.105 - 78153 Le Chesnay Cedex - FRANCE.

[2]Université Orléans la Source, Département de Mathématiques, Campus Universitaire - 45000 Orléans la Source - FRANCE.

[3]Moscow State University, Mechanico-Mathematical Faculty, Chair of probability, Leninskie Gori, 119899 MOSCOW - USSR.

1 Introduction

We consider the problem of explicitly calculating the generating functions of the stationary probabilities for random walks (r.w) in $\mathbf{Z}_+^2$. This problem has at least 20 years history. In a series of papers Malyshev [1,2] suggested some methods and ideas : projection of the functional equation onto some convenient algebraic curve, double Galois group and uniformisation, fundamental theorem about meromorphic continuation.
By a general approach, consisting in a reduction to Riemann-Hilbert boundary value problems on curves in the complex plane, Fayolle and Iasnogorodski [3] derived exact integral representations for some cases. Flatto [4], applying the method of uniformization, obtained an example of algebraic solution.
Here we give necessary and sufficient conditions for rationality and algebraicity of generating functions when the group of the random walk (see below) is finite the latter case is sufficiently large and includes most known queueing theory examples. But in fact the methods are wider and we obtain exact solutions also for non algebraic functions.
First, we recall main facts in sections 2 and 4. Section 3 is devoted to the conditions for the group of the random walk to be finite. In sections 5,6 we give necessary and sufficient conditions for rationality. In section 7 we solve similar question for algebraicity.
The exact sense of our result is the following : we obtain all solutions (algebraic or not) for functional equations on the universal covering, in the class of functions which are meromorphic in the unit circle. We do not specify those of them which have no poles in the unit circles ; with respect to the variables x and y. This is a separate problem which is mainly computational (or even numerical) in a sense which we do not study here.
Our results can be easily applied to abstract Riemann-Hilbert problems for two complex variables (see the formulation of it and some necessary information in [2]), transient behaviour of random walks and some others. In a later paper we shall consider the links between generalized Bessel functions [5] and our methods.

2 Preliminaries

The Random walk

We consider discrete time homogeneous Markov chain $X = \{X_n, \eta \geq 0\}$, with state space $\mathbf{Z}_+^2 = \{(i,j) : i,j \geq 0 \ integers\}$, satisfying the recursive equation

$$X_{n+1} = [X_n + Y_{n+1}]^+ ,$$

where the distribution of Y_{n+1} depends only on the position of X_n in the following way :

$$P\{Y_{n+1} = (i,j)/X_n = (k,l)\} = \begin{cases} p_{ij} & if\ k,l \geq 1, \\ p'_{ij} & k \geq 1,\ l = 0, \\ p''_{ij} & k = 0,\ l \geq 1, \\ p^0_{ij} & k = l = 0. \end{cases}$$

Assumption

- $p_{ij} = 0$ if either $|i| > 1$ or $|j| > 1$, and $p'_{ij}, p''_{ij}, p^0_{ij}$ are zero, if $|z| > d$, or $|j| > d$, for some given $d < \infty$.
- The Markov chain X is irreducible, aperiodic and ergodic [2]. Let $\pi_\alpha = \pi_{kl}$ denote its stationary probabilities.

We introduce the following generating functions

$$\pi(x,y) = \sum_{k,l=1}^{\infty} \pi_{kl}\ x^{k-1}\ y^{l-1}$$

$$\pi(x) = \sum_{k=1}^{\infty} \pi_{k0}\ x^{k-1} \quad ,$$

$$\tilde{\pi}(y) = \sum_{l=1}^{\infty} \pi_{0l}\ y^{l-1}$$

The series are absolutely convergent for $|x|, |y| \leq 1$.
They satisfy the following fundamental equation

$$Q(x,y)\,\pi_(x,y) = q(x,y)\,\pi(x) + \tilde{q}(x,y)\,\tilde{\pi}(y) + q_0(x,y)\,\pi_{00}, \tag{2.1}$$

where $Q, q, \tilde{q}, q_0$ are the polynomials

$$\begin{aligned}
Q(x,y) &= xy(1 - \Sigma\, p_{ij}\, x^i\, y^j) \quad ,\\
q(x,y) &= x(\Sigma\, p'_{ij}\, x^i\, y^j - 1) \quad ,\\
q(x,y) &= y(\Sigma\, p''_{ij}\, x^i\, y^{j-1}) \quad ,\\
q_0(x,y) &= \Sigma\, p^0_{ij}\, x^i\, y^j - 1 \quad .
\end{aligned}$$

We only need elementary facts from Riemann surfaces, algebraic extensions and uniformization (see [9,10]). More precisely, it is known that the equation $Q(x,y) = 0$ defines an algebraic curve of genus 0 or 1, since the degree of Q with respect to each variable x and y is at most 2. We shall make the following.

Assumption $Q(x,y) = 0$ defines an algebraic curve of genus 1. A necessary and sufficient condition for this to be the case was proved in [7]. Basically, they hold for non degenerate random walks, i.e. given any two points (k,l), (k',l'), k,l, $k',l' \geq 1$, one can reach (k',l') starting from (k,l) with a positive probability, without visiting the boundary $\{(i,j)\ : i.j = 0\}$. We give thereafter these conditions.

Lemma 2.1 *For a non degenerate r.w., the genus is 1, iff M_x or M_y is not 0, where*

$$M_x = \sum_{i,j} i p_{ij},\ M_y = \sum_{ij} j p_{ij}\ .$$

In this case, the algebraic curve is non singular and we can identify it with a Riemann surface S for algebraic functions $y(x)$ and $x(y)$. Let s be a point of S. Consider the following covering mappings

$$
\begin{tikzcd}
 & \Omega \arrow[d, "\lambda"] & \\
 & S \arrow[dl, "x(s)"'] \arrow[dr, "y(s)"] & \\
\mathbf{C}_x & & \mathbf{C}_y
\end{tikzcd}
$$

Figure 1

where $\mathbf{C}_x$ is the complex x-plane, $\mathbf{C}_y$ is the complex y-plane, $x(s)$ and $y(s)$ are meromorphic functions on S satisfying

$$Q(x(s),\ y(s)) = 0 \ ,$$

and Ω is a universal covering for S. In the present situation Ω is isomorphic to the complex plane $\mathbf{C}_\omega$, with generic point ω, since the genus of the curve is one. Moreover, $x(\omega)$ and $y(\omega)$ [with the obvious notation introduced in section 4] are elliptic functions.
Let $\mathbf{C}_Q\ (x,y) = \mathbf{C}(S)$ be the field of meromorphic functions on S. It is the algebraic extension of the field $\mathbf{C}(x)$ of rational functions of x and also of course of $\mathbf{C}(y)$. Thus both $\mathbf{C}(x)$ and $\mathbf{C}(y) \subset \mathbf{C}_Q(x,y)$.
The Galois group of $\mathbf{C}_Q(x,y)$ over $\mathbf{C}(x)$ is $\mathbf{Z}/2$ and its generic element ξ is given by

$$\xi(y) = \frac{p_{1,-1}\ x^2 + p_{0,-1}\ x + p_{-1,-1}}{y(p_{11}\ x^2 + p_{01}\ x + p_{-1,1})} \ , \quad \xi(x) = x$$

Analogously the nontrivial element η of the Galois group of $\mathbf{C}_Q(x,y)$ over $\mathbf{C}(y)$ is defined by

$$\eta(x) = \frac{p_{-1,1}\ y^2 + p_{-1,0}\ y + p_{-1,-1}}{x(p_{11}\ y^2 + p_{10}\ y + p_{1,-1})} \ , \quad \eta(y) = y$$

Definition 1 : The group of the random walk is the group of automorphismes of $\mathbf{C}_Q(x,y)$ generated by ξ and η.

3 Explicit conditions, when the group of the random walk is of order 4 or 6

Lemma 3.1 *Let us define $\delta = \eta\xi$.*
Then $\mathcal{H}$ has a normal subgroup $\mathcal{H}_0 = \{\delta^n\}$, which is a cyclic group (finite or infinite) and moreover $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2.
Thus order of $\mathcal{H}$ is thus even (4,6,8,... or ∞).

Now, we shall get simple conditions when the order of the group $\mathcal{H}$ is 4 and 6.

Lemma 3.2 *The order of $\mathcal{H}$ is 4 iff*

$$\begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00}-1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0 \tag{3.1}$$

Proof :
$\delta^2 = I$ is equivalent to $\xi\eta = \eta\xi$ and also entails that
$\eta x \in \mathrm{C}(x)$, $\xi y \in \mathrm{C}(y)$. This means that the automorphisms ξ and η are fractional linear transforms having the form (since $\xi^2 = \eta^2 = 1$),

$$\eta(x) = P(x) = \frac{rx+s}{tx-r}, \;\; \xi(y) = R(y) = \frac{\tilde{r}x+\tilde{s}}{\tilde{t}x-\tilde{r}} \; .$$

Then, we have the following chain of equivalences.
$\eta(x) = P(x) \Longleftrightarrow tx\eta(x) = r(x+\eta(x)) + s$

$\Longleftrightarrow 1,\ x+\eta(x),\ x\eta(x)$ *are linearly dependent*

$\Longleftrightarrow 1, -\frac{b(y)}{a(y)}, \frac{c(y)}{a(y)}$ *are linearly dependent*

$\Longleftrightarrow a(y), b(y), c(y)$ *are also linearly dependent,* where we have put

$$Q(x,y) = a(y)x^2 + b(y)x + c(y)$$

The proof of lemma (3.2) is concluded. ■

Lemma 3.3 *$\mathcal{H}$ has order 6 iff*

$$\begin{vmatrix} \Delta_{23} & \Delta_{33} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & -\Delta_{23} & \Delta_{12} & -\Delta_{22} \\ \Delta_{22} & \Delta_{32} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & -\Delta_{22} & \Delta_{11} & -\Delta_{21} \end{vmatrix} = 0, \tag{3.2}$$

where Δ_{ij} denotes the determinant obtained from (3.1) by suppressing the i-th line and the j-th column.

Proof :
In this case $(\xi\eta)^3 = I$ (identity) which yields

$$\eta\xi\eta = \xi\eta\xi \tag{3.3}$$

Applying (3.3) to x, we get

$$\xi\eta(x) = \eta\xi\eta(x) \ ,$$

which shows that $\xi\eta(x)$ is invariant with respect to η and is thus a rational function of y [since we are working on the algebraic extension of rational fractions]. Similarly, $\eta\xi(y)$ is invariant with respect to ξ and is thus a rational function of x. We have

$$\begin{cases} \xi\eta(x) = P(y) \\ \eta\xi(y) = R(x) \ , \end{cases}$$

which yields

$$y = R(\xi\eta(x)) = R_\circ P(y) \ , \tag{3.4}$$

where $R_\circ P$ is the composition of R *and* P.

Thus $R_\circ P = I$, where it follows easily that P *and* R are indeed linear partial fractions.
From (3.4), we have obtained

$$\xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s} \ , \tag{3.5}$$

which shows that the four elements $1, \xi(y), \eta(x), \xi(y)\eta(x)$ have to be linearly dependent.
Moreover, a simple but tedious computation leads, on the algebraic curve $Q(x,y) = 0$, to the general relationship

$$\begin{cases} \eta(x) &= -\dfrac{xu(y) + v(y)}{xw(y) + u(y)} \ , \\ \\ \xi(y) &= -\dfrac{y\tilde{u}(x) + \tilde{v}(y)}{y\tilde{w}(x) + \tilde{u}(x)} \end{cases} \tag{3.6}$$

with

$$\begin{aligned} u(y) &= \Delta_{22} + y\Delta_{23}, \ \tilde{u}(x) = \Delta_{22} + x\Delta_{32} \ , \\ v(y) &= \Delta_{12} + y\Delta_{13}, \ \tilde{v}(x) = \Delta_{21} + x\Delta_{31} \ , \\ w(y) &= \Delta_{32} + y\Delta_{33}, \ \tilde{w}(x) = \Delta_{23} + x\Delta_{33} \ . \end{aligned}$$

Combining (3.5) and (3.6) leads to the condition (3.2) (the details of the calculus are omitted). The proof of lemma (3.2) is terminated. ■

Examples where $\mathcal{H}$ is of order 4 :

1. Product of 2 independent r.w. inside the quarter plane. Then

$$\sum_{i,j} p_{ij} \, x^i \, y^j = p(x) \, \tilde{p}(y) \ .$$

2. The simple r.w., where $p(x)$ and $\tilde{p}(y)$ are second degree polynomials. Inside the quarter plane,

$$p_{ij} \neq 0 \text{ iff } i \text{ or } j \text{ is zero.}$$

In order words, this would represent independent r.w. in continuous time.

Examples where $\mathcal{H}$ is of order 6

(see fig.2, where only jumps inside the quarter plane have been drawn).

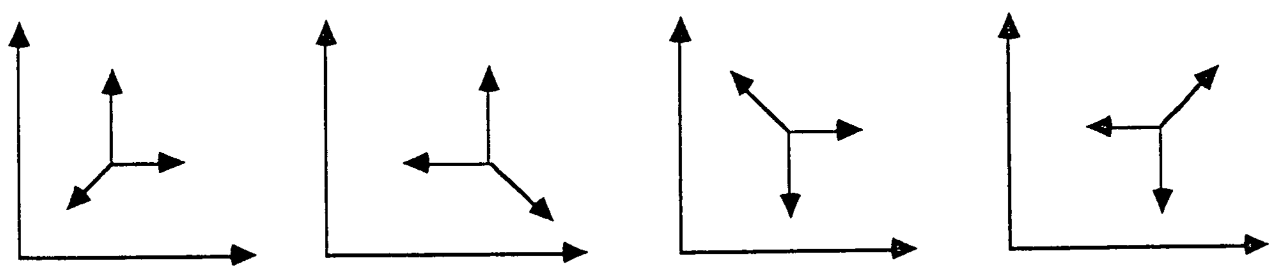

Figure 2

Among these are Flatto's case [4] and Baccelli-Massey's case [5].
The problem of finding the conditions for $\mathcal{H}$ to be finite has many links with some classical problems at algebraic geometry, for instance

1. the problem of abelian integrals of third kind (see [7]);

2. Poncelet's problem [6], pointed out in private discussions by L. Flatto.

4 Solutions of the fundamental equation

Let D and Γ be the unit disk $|z| < 1$ and its boundary $|x| = 1$, in the complex plane $\mathbf{C}$. Let D_x, Γ_x be the inverse images of D, Γ on S under the mapping $x(s) : S \to \mathbf{C}_x$.
Similarly D_y, Γ_y are the inverse images under the mapping $y(s) : S \to \mathbf{C}_y$

Lemma 4.1 *$D_x, D_y, D_x \cup D_y$ are connected domains on S. Γ_x consists of two (closed) curves Γ_0 and $\Gamma_1, \Gamma_0 \cap \Gamma_1 = \phi$. Γ_0 and Γ_1 belong to the same homology class which is one of the generic elements of the torus. Similarly the two components $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ et Γ_y have the same homology class as Γ_0 and Γ_1.*

The inverse image $\lambda^{-1}(D_x \cup D_y)$ consists of a countable number of connected components. We select one of them and shall denote it by Π_0, with the property that $0 \in \Pi_0$. Under the mapping λ^{-1} the functions $x(s)$ *and* $y(s)$ become elliptic functions $x(\omega), y(\omega)$ with periods ω_1, ω_2. We choose the notation so that Π_0 be invariant w.r.t. ω_1. In fact ω_2 is real and ω_1 is pure imaginary but, we do not need this in the sequel of this study.
For the sake of brevity, we shall write

$$q(\omega) = q(x(\omega), y(\omega)) ,$$
$$\pi(\omega) = \pi(x(\omega)), \quad \tilde{\pi}(\omega) = \tilde{\pi}(y(\omega)) \ .$$

From the fundamental equation, we get an equation on Π_0.

$$q(w)\pi(w) + \tilde{q}(w)\tilde{\pi}(w) + q_0(w) = 0 \ . \tag{4.1}$$

(we include π_{00} into q_0)
In fact, equation (4.1) is written first on the domain which is the intersection of the inverse images of D_x *and* D_y, but it is clear that it has a meromorphic continuation to Π_0, since $q, \tilde{q}$ and q_0 are meromorphic functions.

Theorem 4.2 *$\pi(\omega)$ and $\tilde{\pi}(\omega)$ have a meromorphic continuation to the whole $\mathbf{C}_\omega$, where they satisfy (4.1).*

Proof See [7,8]
We introduce the following transformations of $\mathbf{C}_\omega$ into $\mathbf{C}_\omega$, for $\omega \in \mathbf{C}_\omega$.

$$\begin{cases} \eta(\omega) &= \omega_3 - \omega \ , \\ \xi(\omega) &= -\omega \ , \\ \delta(\omega) &= \eta\xi(\omega) = \omega + \omega_3 \ , \end{cases} \tag{4.2}$$

where ω_3 exists and belongs to Π_0,

$$\begin{aligned} (\xi f)(\omega) &= f(\xi(\omega)) \\ (\eta f)(\omega) &= f(\eta(\omega)) \ , \end{aligned} \tag{4.3}$$

for any elliptic function f with periods ω_1, ω_2. It is natural to define the automorphisms ξ *and* η on the field of meromorphic functions on $\mathbf{C}_\omega$ by the same formula (4.3). To prove the theorem, one uses the above translation operators and the property that the interior of the domain $\delta\Pi_0 \cap \Pi_0$ is non empty.
Using equation (4.1), we have the following conditions :

$$\begin{cases} \pi(\omega) = \xi\pi(\omega) = \pi(-\omega) \quad , \\ \tilde{\pi}(\omega) = \eta\tilde{\pi}(\omega) = \tilde{\pi}(\omega_3 - \omega) \ , \\ \pi(\omega + \omega_1) = \pi(\omega) \quad , \\ \tilde{\pi}(\omega + \omega_1) = \tilde{\pi}(\omega) \ . \end{cases} \tag{4.4}$$

Remark : In the next two sections 5 and 6, the fundamental equation is considered on $\mathbf{C}_Q(x, y)$.

5 Rational solutions in the case of finite group and $ff_\delta ... f_{\delta^{n-1}} \neq 1$

Let the group $\mathcal{H}$ of the equation be finite. Then we shall get by strictly algebraic manipulations, rational solutions of the fundamental equation (in the nonhomogeneous case, i.e. when $q_0(x, y) \equiv 0$). These solutions have no (in general) probabilistic meaning, but allow to reduce the problem to the solution of an homogeneous equation.
If we are interested in finding rational solutions we can just consider the main equation

$$Q(x, y)\, \pi(x, q) = \pi(x)q(x, q) + \tilde{\pi}(y)\tilde{q}(x, y) + \pi_{00}\, q_0(x, y) \ ,$$

in $\mathbf{C}_Q(x, y)$, i.e. all calculations are made mod Q. Then, we get immediately

$$q\pi + \tilde{q}\tilde{\pi} + q_0 = 0 \ , \tag{5.1}$$

$$\pi = \pi_\xi \ , \tag{5.2}$$

$$\tilde{\pi} = \tilde{\pi}_\eta \ , \tag{5.3}$$

after having defined, $f_h = hf$, for any automorphism h.
Applying η to (5.1), we get

$$\frac{q_\eta}{\tilde{q}_\eta}\,\pi_\eta + \tilde{\pi} + \frac{q_{0\eta}}{\tilde{q}_\eta} = 0 \tag{5.4}$$

Eliminating $\tilde{\pi}$ from (5.1) and (5.4) yields

$$\frac{q_\eta}{\tilde{q}_\eta}\,\pi_\eta - \frac{q}{\tilde{q}}\,\pi + \frac{q_{0\eta}}{\tilde{q}_\eta} - \frac{q_0}{\tilde{q}} = 0$$

or, since $\pi_\eta = \pi_{\eta\xi} = \pi_\delta$ ($\delta = \eta\xi$ was defined in lemma 3.1),

$$\pi_\delta - f\pi = \psi \ , \tag{5.5}$$

where

$$f = \frac{\varphi}{\varphi_\eta},\ \varphi = \frac{q}{\tilde{q}}\ , \tag{5.5)bis}$$

$$\psi = \frac{1}{\varphi_\eta}(F - F_\eta),$$

$$F = \frac{q_0}{\tilde{q}}\,\pi_{00}\ .$$

Upon applying δ respeatedly in (5.5), we obtain the system of equations :

$$\begin{array}{l} \pi_\delta - f\pi = \psi\ , \\ \pi_\delta - f_\delta\,\pi_\delta = \psi_\delta\ , \\ \dots\dots\dots\dots \\ \pi_{\delta^n} - f_{\delta^{n-1}}\ \pi_{\delta^{n-1}} = \psi_{\delta^{n-1}} \end{array} \tag{5.6}$$

If $\mathcal{H}$ is of order n, then this system is closed and $\pi_{\delta^n} = \pi$ (as $\delta^n = id$). It is possible to find an explicit expression for π. To this end, let us multiply each equation, starting from the $(n-1) - th$ on, by

$$f_{\delta^{n-1}},\ f_{\delta^{n-1}}\ f_{\delta^{n-2}}, \dots,\ f_{\delta^{n-1}}\ f_{\delta^{n-2}} \dots f_\delta\ ,$$

respectively.

Then, summing up all these equations under the assumption

$$f f_\delta \, f_{\delta 2} \ldots f_{\delta^{n-1}} \neq 1 \quad , \tag{5.7}$$

we get

$$\pi = \frac{\psi_{\delta^{n-1}} + \psi_{\delta^{n-2}} f_{\delta^{n-1}} + \ldots + \psi f_{\delta^{n-1}} f_{\delta^{n-2}} \ldots f_\delta}{1 - f f_\delta f_{\delta 2} \ldots f_{\delta^{n-1}}} \tag{5.8}$$

Similarly, if we put

$$\delta^{-1} = \tilde{\delta} = \xi\eta \quad ,$$

$$\tilde{\psi} = \frac{q_\xi}{\tilde{q}_\xi}\left(\frac{q_0}{q} - \frac{(q_0)_\xi}{q_\xi}\right) = \frac{1}{\varphi_\xi}(F - F_\xi)$$

$$\tilde{f} = \frac{\varphi}{\varphi_\xi} \quad ,$$

then, under the condition,

$$\tilde{f} \tilde{f}_\delta \ldots \tilde{f}_{\delta^{n-1}} \neq 1 \quad , \tag{5.9}$$

we get

$$\tilde{\pi} = \frac{\tilde{\psi}_{\delta^{n-1}} + \tilde{\psi}_{\delta^{n-2}} \tilde{f}_{\tilde{\delta}^{n-1}} + \ldots + \tilde{\psi} \tilde{f}_{\tilde{\delta}^{n-1}} \tilde{f}_{\delta^{n-2}} \ldots \tilde{f}_\delta}{1 - \tilde{f} \tilde{f}_\delta \ldots \tilde{f}_{\tilde{\delta}^{n-1}}} \tag{5.10}$$

Let us prove that conditions (5.7) and (5.9) are in fact equivalent. Some definitions from algebra are now necessary.
Let us denote by $\mathbf{C}_\delta(x,y)$ the subfield of $\mathbf{C}_Q(x,y)$, of all elements which are invariant w.r.t. δ. For $f \in \mathbf{C}_Q(x,y)$ we can define the **trace** and the **norm** :

$$Tr_{\mathbf{C}_\delta}^{\mathbf{C}_Q}(f) = \sum_{h \in \mathcal{H}_0} hf = f + f_\delta + \ldots + f_{\delta^{n-1}} \quad ,$$

$$N(f) = N_{\mathbf{C}_\delta}^{\mathbf{C}_Q}(f) = \prod_{h\in\mathcal{H}_0} hf = ff_\delta ... f_{\delta^{n-1}} \quad ,$$

where n is the order of the cyclic group $\mathcal{H}_0 = \{\delta^k,\ k \geq 0\}$. We have

$$N(f) = N(\varphi)N(\varphi_\delta^{-1}) = \frac{N\varphi}{N(f_\delta)}, N(\tilde{f}) = N(\varphi^{-1})N(\varphi_\xi) \quad .$$

Moreover it is easy to check that $N(\varphi_\xi) = N(\varphi_\delta)$. Thus

$$N(f) = \frac{N(\varphi)}{N(\varphi_\delta)} = \frac{N(\varphi)}{N(\varphi_\xi)} = \frac{1}{N(\tilde{\varphi})} \,. \tag{5.11}$$

Theorem 5.1 *Let the order of $\mathcal{H}$ be $2n$ and condition (5.7) holds. Then there is a rational solution of (5.1), (5.2), (5.3). Moreover, π and $\tilde{\pi}$ are given by (5.8) and (5.10) respectively.*

Proof. We have just proved uniqueness.

To prove existence, one must only prove that $\pi = \pi_\xi$. For this we use

$$ff_\eta = 1,\ \frac{\psi}{\psi_\eta} = -f,\ \xi\delta^{n-i} = \delta^{i+1}\eta \tag{5.12}$$

So we get

$$\psi_{\xi\delta^{n-i}} = \psi_{\delta^{i+1}\eta} = -\frac{\psi_{\delta^{i+1}}}{f_{\delta^{i+1}}} \tag{5.13}$$

$$f_{\xi\delta^{n-i+1}} = \frac{1}{f_{\delta^i}}$$

and

$$\pi_\xi = \frac{(\Sigma_{i=1}^n \psi_{\delta^{n-i}} \Pi_{k=2}^i f_{\delta^{n-k+1}})_\xi}{1 - \Pi_{i=0}^{n-1} \frac{1}{f_{\delta^i}}} = \frac{-\Sigma_{i=1}^n \frac{\psi_{\delta^{i+1}}}{f_{\delta^{i+1}}} \prod_{k=2}^i \frac{1}{f_{\delta^k}}}{1 - \prod_{i=0}^{n-1} \frac{1}{f_{\delta^i}}} = \pi \,.$$

The proof of theorem (5.1) is concluded. ∎

As already mentioned earlier, such rational solutions night have no probabilistic significance, due to the possible existence of poles in the unit disk. Therefore, we must search for all the solutions.

A constructive algorithm to compute π *and* $\tilde{\pi}$,

From (5.8) and (5.10), we can get particular solutions π and $\tilde{\pi}$ as rational functions of x and y

$$\pi = \frac{P_0(x,y)}{P_1(x,y)} \ , \ \tilde{\pi} = \frac{Q_0(x,y)}{Q_1(x,y)} \ , \tag{5.14}$$

where $P_i(x,y)$, $Q_i(x,y)$ are polynomials of y with coefficients in $\mathbf{C}(x)$, i.e. in the field of rational functions of x. One can divide both the numerator and the denominator of (5.14) by the polynomial of y

$$Q(x,y) = a(x)y^2 + b(x)y + c(x)$$

and get, modulo Q,

$$\pi = \frac{A_1(x)y + A_0(x)}{B_1(x)y + B_0(x)} \ .$$

But, we have proved that π is invariant w.r.t. ξ. Then, from the main theorem of Galois theory, it follows that

$$\frac{A_1(x)y + A_0(x)}{B_1(x)y + B_0(x)} = \phi(x) \ mod \ Q(x,y) \ , \tag{5.15}$$

where $\phi(x) \in \mathbf{C}(x)$.

Equation (5.15) can be rewritten as

$$[A_1(x) - B_1(x)\phi(x)]y + A_0(x) - B_0(x)\phi(x) = 0 \ mod \ Q(x,y) \ . \tag{5.16}$$

But (5.16) is possible only if

$$A_0(x) = B_0(x)\phi(x) \ and \ A_1(x) = B_1(x)\phi(x) \ ,$$

whence

$$\pi = \phi(x) \ .$$

6 Rational solutions in the case of finite group and $ff_\delta ... f_{\delta^{n-1}} = 1$

In section 5, it was shown that, whenever condition (5.7) takes place,

$$\pi = w + R\,, \tag{6.1}$$

where $R \in \mathrm{C}(x)$ is rational [given by Theorem 5.1] and w satisfies the homogeneous equation

$$w_\delta - fw = 0\ . \tag{6.2}$$

When

$$ff_\delta ... f_{\delta^{n-1}} = 1\ \ ,$$

we shall now prove that

$$\pi = aw\ \ ,$$

where $a \in \mathrm{C}(x)$ is rational and w satisfies the equation

$$w_\delta - w = a_\delta \psi\ \ .$$

"En passant", we shall also find the possible rational solutions.

Lemma 6.1 *Let F_h, F be two fields such that*
i) F is a finite algebraic extension of F_h.
ii) The Galois group of automorphisms $\mathcal{H}$ defined on F is cyclic and generated by some element h.

$$\mathcal{H} = \{h^k\}\ \ .$$

Then, for $\phi, \psi \in F$ such that

$$N_{F_h}^F(\phi) = 1\ \ ,$$

$$Tr^F_{F_h}(\psi) = 0 \ ,$$

(the norm and the trace have been defined in section 5) there exist $a, b, \in F$ satisfying respectively

$$\begin{aligned} \phi &= \frac{a}{a_h} && (6.3) \\ \psi &= b - b_h && (6.4) \end{aligned}$$

Proof :

The result follows directly from the additive (resp. multiplicative) form of Hilbert's theorem 90 [see 9, p.213], which ensures moreover that there exists $\theta \in F$, such that

$$a = \theta + \phi\theta_h + \phi\phi_h\,\theta_{h^2} + \ldots + \phi\phi_h\ldots\phi_{h^{n-2}}\,\theta_{h^{n-1}} \ ,$$

where the right member is not identically zero.

Similarly, for any $\gamma \in F$ such that

$Tr^F_{F_h}(\gamma) \neq 0$, (for example $\gamma \in F_h - \{0\}$),

$$b = \frac{1}{Tr^F_{F_h}(\gamma)}\,[\psi\gamma + (\psi + \psi_h)\gamma_h \ldots + (\psi + \ldots\psi_{h^{n-2}})\gamma_{h^{n-2}}]$$

[This assertions can be checked by simple computations] ■

Lemma 6.2 *Let us take $h \equiv \delta = \eta\xi(\eta^2 = \xi^2 = I)$ and $f \in F$ in the preceding lemma, such that*

$$\begin{cases} N^F_{F_h}(f) = 1 \\ N^F_{F_\eta}(f) = ff_\eta = 1 \ . \end{cases}$$

Then there exists $c \in F_\xi$, satisfying

$$f = \frac{c}{c_\delta} \ .$$

Proof :

From Lemma 6.1, we know that there exits $a \in F$, with

$$f = \frac{a}{a_\delta} \ .$$

Hence

$$f_\eta = \frac{a_\eta}{a_\xi} = \frac{1}{f} = \frac{a_\delta}{a} \ ,$$

which yields in turn

$$\frac{a_\eta}{a_\delta} = \frac{a_\xi}{a} \ . \tag{6.5}$$

But

$$\frac{a_\eta}{a_\delta} = (\frac{a}{a_\xi})_\eta \ .$$

Thus (6.5) can be rewritten as

$$(\frac{a}{a_\xi})_\xi = (\frac{a}{a_\xi})_\eta \ ,$$

which is equivalent to

$$(\frac{a}{a_\xi})_\delta = \frac{a}{a_\xi} \ .$$

Let us introduce the sub-field

$$F_{\xi,\eta} = F_\xi \bigcap F_\eta \ .$$

F_δ is an algebraic extension of degree 2 of $F_{\xi,\eta}$ generated by ξ (*or* η) with a group of automorphisms $(1, \xi)$.
We again apply lemma (4.1) to the function

$$\frac{a_\xi}{a}, \quad since \quad N^{F_\delta}_{F_{\xi,\eta}}(\frac{a_\xi}{a}) = 1 \ .$$

Hence, there exists $b \in F_\delta$ such that

$$\frac{b}{b_\xi} = \frac{a_\xi}{a} \ ,$$

which entails that $ab \in F_\xi$.
Putting $c = ab$, we have,

$$\frac{c}{c_\delta} = \frac{ab}{a_\delta b_\delta} = \frac{a}{a_\delta} = f$$

The proof of lemma 6.2 is terminated. ■

Lemma 6.3 *The notation is the same as in lemmas 6.1 and 6.2. Let us take $u \in F$ such that*

$$\begin{cases} Tr^F_{F_h}(u) = 0 \ , \\ \\ Tr^F_{F_\eta}(u) = 0 \ . \end{cases}$$

Then, there exists $\gamma \in F_\xi$, satisfying $u = \gamma - \gamma_\delta$.

Proof :
The arguments rely on the additive Hilbert's theorem 90. From the assumption, there exists $l \in F$, such that $l - l_\delta = u$.
Since $(l_\delta - l)_\eta + (l_\delta - l) = 0$,
we have

$$l_\xi - l_\eta + l_\delta - l = 0 \ ,$$

or

$$l - l_\xi = l_\delta - l_\eta = (l - l_\xi)_\delta$$

But

$$Tr^{F_\delta}_{F_{\xi,\eta}}(l - l_\xi) = 0 \ .$$

Thus, there exists $v \in F_\delta$ such that

$$v - v_\xi = l - l_\xi$$

Choosing

$$\gamma = v - l \ ,$$

concludes the proof of the lemma. ■

In the applications of lemmas 6.1, 6.2, 6.3 in this section, we choose $F = \mathbf{C}_Q$; then $F_\xi = \mathbf{C}(x)$, $F_\eta = \mathbf{C}(y)$. We are now in a position to formulate the final result of this section.

Theorem 6.4 *Let $N(f) = 1$.*
Then the equation (5.5) has a rational solution iff

$$Tr(c_\delta \psi) = 0 \ , \tag{6.6}$$

where $c \in \mathbf{C}(x)$ will be defined below. Moreover under this condition this solution is unique up to solutions of the equations

$$\pi_\xi = \pi_\eta = \pi$$

Proof :
Coming back to the basic equation (5.5), we get, from lemma 6.2,

$$(c\pi)_\delta - c\pi = c_\delta \psi \ , \tag{6.7}$$

where $c \in \mathbf{C}(x)$.
Thus (5.5) has been reduced to the following equation

$$w_\delta - w = c_\delta \psi \ , \tag{6.8}$$

where $c_\delta \psi \in C_Q(x, y)$.
A straightforward computation yields

$$c_\delta \psi = g_\eta - g \ ,$$

where

$$g = -\frac{cF}{\varphi}$$

Thus, (6.7) can be rewritten as

$$w_\delta - w = w_\eta - w = g_\eta - g \ , \tag{6.9}$$

where we have used the relationship $w_\xi = w$.

The function $c_\delta\psi = g_\eta - g$ satisfies the second condition of lemma 6.3. If it also satisfies the condition

$$Tr(c_\delta\psi) = 0 \ .$$

Hence, lemma 6.3 ensures the existence of $\gamma \in \mathbf{C}(x)$, such that

$$c_\delta\psi = \gamma - \gamma_\delta \ ,$$

and equation (6.8) becomes

$$w_\delta - w = \gamma - \gamma_\delta$$

or

$$(w+\gamma)_\delta = w + \gamma \ .$$

Thus

$$w = -\gamma + u \ ,$$

where u satisfies $u_\delta = u$.
Thus rational solutions of (6.7) are given by

$$\pi = \frac{-\gamma + u}{c} \ ,$$

where u is a rational solution of the system of equations

$$u = u_\xi = u_\eta \ .$$

On the other hand, if (6.7) has a rational solution, then equation (6.8) also has a rational solution w and, since $Tr(w - w_\delta) = 0$, $Tr(C_\delta\psi) = 0$.
The proof of theorem 6.1 is terminated. ■

7 Algebraic solutions in the case of finite group

Let $\mathcal{H}$ be of order $n < \infty$. Consider, on the universal covering, the equation

$$\pi_\delta - f\pi = \psi \ , \tag{7.1}$$

where we mainly use theorem 4.2, and the relations [see (5.5)bis]

$$f = \frac{\varphi}{\varphi_\eta},\ \psi = \frac{1}{\varphi_\eta}(F - F_\eta)\,.$$

Thus, we are looking for solutions of (7.1) in the class of functions

$$\pi(\omega) = \pi(-\omega)\ . \tag{7.2}$$

We must consider separately the cases

$$N(f) = 1 \ \ and \ \ N(f) \neq 1.$$

Case I $N(f) = 1$.
By Lemma 6.2, there exists $c \in \mathbb{C}(x)$ such that

$$f = \frac{c}{c_\delta}\ .$$

Theorem 7.1 *Equations (7.1) and (7.2) have a non zero algebraic solution iff*

$$Tr(c_\delta \psi) = 0$$

or, equivalently,

$$\sum_{k=0}^{n-1} \psi_{\delta^{k+1}} \prod_{i=k}^{n-1} f_{\delta^i} = 0\ . \tag{7.3}$$

Any such solution is of the form

$$\pi = \pi_0 + \frac{w}{c}\ , \tag{7.4}$$

where π_0 is some fixed solution and w is an algebraic solution of the equations

$$w_\eta = w_\xi = w\ . \tag{7.5}$$

According to custom, by "algebraic" we mean that w is moromorphic on $\mathbf{C}_\omega$ and there exists some polynomial P satisfying

$$P(w(\omega),\ x(\omega)) \equiv 0\ . \tag{7.6}$$

Proof : Substituting $f = \frac{c}{c_\delta}$ in (7.1), we get

$$(c\pi)_\delta - c\pi = c_\delta\psi\ .$$

If $Tr(c_\delta\psi) = 0$, then

$$c_\delta\psi = \gamma - \gamma_\delta,\ \gamma \in \mathbf{C}(x)\ ,$$

by lemma 6.3. Thus

$$(c\pi + \gamma)_\delta = c\pi + \gamma\ .$$

Denoting

$$w = c\pi + \gamma\ ,$$

we obtain

$$\pi = -\frac{\gamma}{c} + \frac{w}{c}$$

and

$$w_\delta = w = w_\xi\ .$$

This means that w is meromorphic in $\mathbf{C}_\omega$ and is an elliptic function with periods ω_1, ω_3. But the group $\mathcal{H}$ is finite, whence

$$\omega_3 = \frac{k_1}{n}\omega_1 + \frac{k_2}{n}\omega_2\ ,$$

for some integers k_1, k_2, n. It follows, see [11], that (7.6) holds for some polynomial P.
Let now $Tr(c_\delta\psi) \neq 0$. We shall prove that (7.1), (7.2) have no algebraic solutions. In this case,

$$c_\delta\psi = \psi_1 + \psi_2\ , \tag{7.7}$$

where

$$Tr(\psi_1) = 0,\ (\psi_2)_\delta = \psi_2\ .$$

Indeed (7.7) is easily derived by putting

$$\begin{cases} \psi_2 = \frac{1}{n}\, Tr(c_\delta \psi) \ , \\ \psi_1 = c_\delta \psi - \frac{1}{n}\, Tr(c_\delta \psi) \ . \end{cases}$$

In this case all solutions of (7.1), (7.2) have the form

$$w = w_1 + w_2 + U \ ,$$

where w_1 is rational and given by Hilbert's decomposition

$$\psi_1 = w_1 - (w_1)_\delta \ ,$$

w_2 is a particular solution of

$$(w_2)_\delta - w = \psi_2$$

and U is an algebraic function with the properties

$$U_\delta = U_\xi = U \ .$$

All w_1, w_2, U are invariant w.r.t. ξ.
To find w_2, let us consider the equation

$$w_\delta - w = \psi_2$$

on the universal covering. We shall show that $w_2(\omega) = \psi_2(\omega)\Phi(\omega)$, where

$$\Phi(\omega) = \frac{\omega_1}{2\pi_i}\, \zeta(\omega; \omega_1, \omega_3) - \frac{\omega}{\pi_i}\zeta(\frac{\omega_1}{2}; \omega_1, \omega_3) \ .$$

where $\zeta(\omega)$ is the classical "zeta function".

Lemma 7.2 *ψ_2 is odd, i.e.* $\psi_2(\omega) = -\psi_2(-\omega)$.

Proof : Since

$$\psi_2 = \frac{1}{n}\, Tr(c_\delta \psi)$$

and (see preceding paragraph) there exists g such that

$$c_\delta \psi = g_\eta - g \ ,$$

we have

$$\psi_2 = \frac{1}{n}\, Tr(g_\eta) - \frac{1}{n}\, Tr(g) \ .$$

But $\delta_\eta^i = \eta(\delta^{-1})^i = \eta\delta^{n-i}$, which yields

$$Tr\ (g_\eta) = (Tr(g))_\eta \ .$$

It follows that

$$\psi_2 = \frac{1}{n}\, [Tr(g_\eta) - Tr(g)]$$

and

$$\psi_2 + (\psi_2)_\eta = 0 \ .$$

As ψ_2 is invariant w.r.t. δ, we also have

$$\psi_2 + (\psi_2)_\xi = 0 \ .$$

Lemma 7.2 is proved. ■

Lemma 7.3 *The function $\Phi(w)$ introduced in the proof of theorem 7.1 is meromorphic in $\mathbf{C}_\omega$, odd and periodic with period ω_1 and satisfies*

$$\Phi(\omega + \omega_3) = \Phi(\omega) + 1 \ .$$

Proof : Φ is meromorphic and odd since ζ has both these properties. To prove the second assertion we use

$$\zeta(\omega + \omega_i) = \zeta(\omega) + 2\zeta(\frac{\omega_i}{2}),\ i = 1, 3$$

and Legendre's identity [ ]

$$\omega_1\zeta(\frac{\omega_3}{2}) - \omega_3\zeta(\frac{w_1}{2}) = \pi i \ .$$

Lemma 7.3 is proved. ■

From these two lemmas, it follows that the product $\psi_2\Phi$ is meromorphic, periodic with period ω_1, even and

$$(\psi_2\Phi)_\delta(\omega) = [(\psi_2)_\delta\ \Phi_\delta](\omega) =$$
$$\psi_2(\omega)\Phi(\omega+\omega_3) = \psi_2(\omega)\Phi(\omega) + \psi_2(\omega) \ .$$

Thus $\psi_2\Phi$ is a solution of the equation

$$w - w_\delta = \psi_2 \ .$$

As w_1 *and* U are algebraic, w is algebraic iff w_2 is algebraic. It remains to show that Φ is not algebraic. To that end, we use the fact that Φ has a linear growth in the ω_3-direction. Introducing, for all integers m (positive or negative) and a fixed ω,

$$\omega_m = \omega + m(k_1\omega_1 + k_2\omega_2) = \omega + mn\omega_3 \ ,$$

we have

$$x = x(\omega) = x(\omega_m) \ , \forall m \ \in \mathbf{Z}_+ \ .$$

Consequently,

$$\Phi(\omega_m) = \Phi(\omega) + mn \ , \ \forall m \ \in \mathbf{Z}_+ \ .$$

Thus Φ, for the same x, takes an infinite number of values and cannot be algebraic in x.
The proof of theorem 7.1 is terminated. ■

Case II $N(f) \neq 1$

Theorem 7.4 *Equation (7.1), (7.2) have an algebraic nonrational solution iff*

$$N(f) = -1$$

Proof : Since $N(f) \neq 1$, then (7.1), (7.2) have a rational solution. Hence, we can consider only the homogeneous equation (7.1).
Let us suppose that the equations

$$\begin{cases} \pi_\delta = f\pi \ , \\ \pi_\xi = \pi \ , \end{cases} \tag{7.8}$$

where f satisfies

$$ff_\eta = 1 \ ,$$

have an algebraic solution, i.e. $P(\pi, x) \equiv 0$ for some polynomial P.

Notation : For any function $g : t \mapsto g(t)$, we shall set $g'_t \equiv \frac{dg}{dt}$, the derivative of g w.r.t. t.
By the theorem on implicit functions, we know that π'_x is rational of π and x. Hence the ratio $\frac{\pi'_x}{\pi}$ is algebraic in x.
Moreover, since $x(\omega)$ is an elliptic function of ω, $x'_\omega(\omega)$ is also elliptic with the same periods. Thus x'_ω and $\frac{\pi'\omega}{\pi}$ are algebraic of x, after using the elementary equality

$$\frac{\pi'_\omega}{\pi} = \frac{\pi'_x}{\pi}\, x'_\omega \ .$$

Define the following logarithmic derivatives

$$u = \frac{\pi'}{\pi} \quad and \quad v = \frac{f'}{f} \ .$$

By using, (7.8), we get

$$\begin{cases} u_\delta - u = v \\ u_\xi + u = 0 \\ v_\eta - v = 0 \end{cases} \tag{7.9}$$

Introduce, temporarily, the following automorphism $\hat{\xi}, \hat{\eta}$ (under the same conditions as for ξ *and* η)

$$(\hat{\xi}h)(\omega) = -h(\xi(\omega)) \ ,$$
$$(\hat{\eta}h)(\omega) = -h(\eta(\omega)) \ ,$$

for any meromorphic function h on the universal covering.

Rewriting (7.9) with the notation $\hat{\xi}, \hat{\eta}$ and remarking that $\delta = \eta\xi = \hat{\eta}\hat{\xi}$, yields

$$
\begin{aligned}
u_{\delta} - u &= v \ , \\
u_{\hat{\xi}} - u &= 0 \ , \\
v_{\hat{\eta}} + v &= 0 \ .
\end{aligned}
$$

Now, we are in a position to apply the theorem 7.1, simply replacing in its statement f by 1 and ψ by v. Thus,

$$Tr(v) = 0 \ .$$

But

$$Tr(v) = Tr(\frac{f'}{f}) = \frac{N(f)'}{N(f)} \ ,$$

since

$$\delta^k(\omega) = \omega + k\omega_3 \ ,$$

which in turn implies

$$\frac{d\delta^k(\omega)}{d\omega} = 1 \ .$$

Then $Tr(v) = 0$ implies $N(f) = K$.
We have the following chain of equalities

$$
\begin{aligned}
& f.f_{\eta} = 1 \Longrightarrow N(f).N(f_{\eta}) = 1 \Longrightarrow N(f).(N(f))_{\eta} = 1 \Longrightarrow K^2 = 1 \\
& \Longrightarrow K = -1 \ , \ since \ K \neq 1 \ .
\end{aligned}
$$

So we proved the "if" assertion of the theorem.
Now, let us consider the homogeneous equation with $N(f) = -1$. We want to prove that this equation has an algebraic solution. For this, let us note that

$$\prod_{i=0}^{2n-1} f_{\delta^i} = (\prod_{i=0}^{n-1} f_{\delta^i})^2 = (N(f))^2 = 1 \ ,$$

as $f \in \mathbf{C}_Q(x, y)$ and $\delta^n = id$.

We would like to get a factorization of f (like Hilbert factorisation). For this, let us define the field $\mathcal{K}$ of elliptic functions with periods $2n\omega_3, \omega_1$.
$\mathbf{C}_Q(x,y)$ can be considered as a subfield of this field. If $u \in \mathcal{K}$, then δ is defined, using 4.2, by

$$u_\delta(\omega) = u(\omega + \omega_3) \ .$$

The cyclic group generated by δ on $\mathcal{K}$ is finite of order $2n$. It is clear that $f \in \mathcal{K}$ and

$$N^{\mathcal{K}}_{\mathcal{K}_\delta}(f) = 1 \ ,$$

where $\mathcal{K}_\delta$ is the subfield of elliptic functions with periods ω_3 and ω_1.
By Lemma 6.2, there exists $c \in \mathcal{K}_\xi$ (the subfield of elements of $\mathcal{K}$, invariant w.r.t. ξ), such that

$$f = \frac{c}{c_\delta} \ ,$$

since

$$N^{\mathcal{K}}_{\mathcal{K}_\eta}(f) = f.f_\eta = 1 \ .$$

It follows

- first, that c is an even elliptic function with periods ω_1 and $2n\omega_3 = 2k\omega_2$;
- secondly that x, which is an elliptic function with primitive periods ω_1 and ω_2, is related to c by a polynomial relationship.
  The proof of theorem 7.4 is concluded. ■

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